

# Families of Thue equations associated with a rank one subgroup of the unit group of a number field

by

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ABSTRACT. Twisting a binary form  $F_0(X, Y) \in \mathbb{Z}[X, Y]$  of degree  $d \geq 3$  by powers  $v^a$  ( $a \in \mathbb{Z}$ ) of an algebraic unit  $v$  gives rise to a binary form  $F_a(X, Y) \in \mathbb{Z}[X, Y]$ . More precisely, when  $K$  is a number field of degree  $d$ ,  $\sigma_1, \sigma_2, \dots, \sigma_d$  the embeddings of  $K$  into  $\mathbb{C}$ ,  $\alpha$  a nonzero element in  $K$ ,  $a_0 \in \mathbb{Z}$ ,  $a_0 > 0$  and

$$F_0(X, Y) = a_0 \prod_{i=1}^d (X - \sigma_i(\alpha)Y),$$

then for  $a \in \mathbb{Z}$  we set

$$F_a(X, Y) = a_0 \prod_{i=1}^d (X - \sigma_i(\alpha v^a)Y).$$

Given  $m \geq 0$ , our main result is an effective upper bound for the solutions  $(x, y, a) \in \mathbb{Z}^3$  of the Diophantine inequalities

$$0 < |F_a(x, y)| \leq m$$

for which  $xy \neq 0$  and  $\mathbb{Q}(\alpha v^a) = K$ . Our estimate involves an effectively computable constant depending only on  $d$ ; it is explicit in terms of  $m$ , in terms of the heights of  $F_0$  and of  $v$ , and in terms of the regulator of the number field  $K$ .

KEYWORDS: Families of Thue equations, Diophantine equations

MSC: 11D61, 11D41, 11D59

## 1 Introduction and the main results

Let  $d \geq 3$  be a given integer. We denote by  $\kappa_1, \kappa_2, \dots$  positive effectively computable constants which depend only on  $d$ .

Let  $K$  be a number field of degree  $d$ . Denote by  $\sigma_1, \sigma_2, \dots, \sigma_d$  the embeddings of  $K$  into  $\mathbb{C}$  and by  $R$  the regulator of  $K$ . Let  $\alpha \in K$ ,  $\alpha \neq 0$ , and let  $a_0 \in \mathbb{Z}$ ,  $a_0 > 0$ , be such that the coefficients of the polynomial

$$f_0(X) = a_0 \prod_{i=1}^d (X - \sigma_i(\alpha))$$

are in  $\mathbb{Z}$ . Let  $v$  be a unit in  $K$ , not a root of unity. For  $a \in \mathbb{Z}$ , define the polynomial  $f_a(X)$  in  $\mathbb{Z}[X]$  and the binary form  $F_a(X, Y)$  in  $\mathbb{Z}[X, Y]$  by

$$f_a(X) = a_0 \prod_{i=1}^d (X - \sigma_i(\alpha v^a))$$

and

$$F_a(X, Y) = Y^d f_a(X/Y) = a_0 \prod_{i=1}^d (X - \sigma_i(\alpha v^a) Y).$$

Define

$$\lambda_0 = a_0 \prod_{i=1}^d \max\{1, |\sigma_i(\alpha)|\} \quad \text{and} \quad \lambda = \prod_{i=1}^d \max\{1, |\sigma_i(v)|\}.$$

Let  $m \in \mathbb{Z}$ ,  $m > 0$ . We consider the family of Diophantine inequalities

$$(1) \quad 0 < |F_a(x, y)| \leq m,$$

where the unknowns  $(x, y, a)$  take their values in the set of elements in  $\mathbb{Z}^3$  such that  $xy \neq 0$  and  $\mathbb{Q}(\alpha v^a) = K$ . It follows from the results in [4] that the set of solutions is finite. However, the proof in [4] relies on Schmidt's subspace theorem, which is not effective. Here we give an effective upper bound for  $\max\{|x|, |y|, |a|\}$  in terms of  $m$ ,  $R$ ,  $\lambda_0$  and  $\lambda$ , by using lower bounds for linear forms in logarithms.

For  $x \in \mathbb{R}$ ,  $x > 0$ , we stand to the notation  $\log^* x$  for  $\max\{1, \log x\}$ .

Here is our main result.

**Theorem 1.** *There exists an effectively computable constant  $\kappa_1 > 0$ , depending only on  $d$ , such that any solution  $(x, y, a) \in \mathbb{Z}^3$  of (1), which verifies  $xy \neq 0$  and  $\mathbb{Q}(\alpha v^a) = K$ , satisfies*

$$|a| \leq \kappa_1 \lambda^{d^2(d+2)/2} (R + \log m + \log \lambda_0) R \log^* R.$$

Under the assumptions of Theorem 1, with the help of the upper bound

$$H(F_a) \leq 2^d \lambda_0 \lambda^{|a|}$$

for the height of the form  $F_a$ , it follows from the bound (3.2) in [1, Theorem 3] (see also [2, Th. 9.6.2]) that

$$\log \max\{|x|, |y|\} \leq \kappa (R + \log^* m + |a| \log \lambda + \log \lambda_0) R (\log^* R)$$

with

$$\kappa = 3^{r+27} (r+1)^{7r+19} d^{2d+6r+15}.$$

Combining this upper bound with our Theorem 1 provides an effective upper bound for  $\max\{|x|, |y|, |a|\}$ .

For  $i = 1, \dots, d$ , set  $v_i = \sigma_i(v)$  and assume

$$|v_1| \leq |v_2| \leq \dots \leq |v_d|.$$

Our proof actually gives a much stronger estimate for  $|a|$ , see Theorem 2, which involves some extra parameter  $\mu > 1$  defined by

$$\mu = \begin{cases} \lambda & \text{if } |v_1| = |v_{d-1}| \text{ or } |v_2| = |v_d|, \\ \min \left\{ \frac{|v_{d-1}|}{|v_1|}, \frac{|v_d|}{|v_2|} \right\} & \text{if } |v_1| < |v_2| = |v_{d-1}| < |v_d|, \\ \frac{|v_{d-1}|}{|v_2|} & \text{if } |v_2| < |v_{d-1}|. \end{cases}$$

Notice that the condition  $|v_1| = |v_{d-1}|$  means  $|v_1| = |v_2| = \dots = |v_{d-1}|$  and that the condition  $|v_2| = |v_d|$  means  $|v_2| = |v_3| = \dots = |v_d|$ ; using Lemma 11, we deduce that each of these two conditions implies that  $d$  is odd, hence that the field  $K$  is almost totally imaginary (namely, with a single real embedding) – compare with [9].

**Theorem 2.** *There exists a positive effectively computable constant  $\kappa_2$ , depending only on  $d$ , with the following property. Let  $(x, y, a) \in \mathbb{Z}^3$  satisfy*

$$xy \neq 0, \quad [\mathbb{Q}(\alpha v^a) : \mathbb{Q}] = d \quad \text{and} \quad 0 < |F_a(x, y)| \leq m.$$

*Then*

$$(2) \quad |a| \leq \kappa_2 \frac{\log \lambda}{\log \mu} (R + \log m + \log \lambda_0 + \log \lambda) R \log \left( R \frac{(\log \lambda)^2}{\log \mu} \right).$$

On the one hand, using Lemma 12 (§3.5), we will prove in §5 that

$$\log \mu \geq \kappa_3 \lambda^{-d^2(d+2)/2} (\log \lambda)^2,$$

which will enable us to deduce Theorem 1 from Theorem 2. On the other hand, thanks to (5), we have  $\mu \leq \lambda^2$ . In general, we expect  $\mu$  to be as large as  $\lambda^{\kappa_4}$  (which is therefore the maximum possible), in which case the conclusion of Theorem 2 becomes

$$(3) \quad |a| \leq \kappa_5 (R + \log m + \log \lambda_0 + \log \lambda) R (\log R + \log^* \log^* \lambda)$$

with a positive effective constant  $\kappa_5$  depending only on  $d$ . In §2, we give a few examples where this last bound is valid.

In Theorem 1, the hypothesis that  $v$  is not a root of unity cannot be omitted. Here is an example with  $\alpha = a_0 = m = 1$ . Let  $\Phi_n(X)$  be the cyclotomic polynomial of index  $n$  and degree  $\varphi(n)$  (Euler totient function).

Let  $\zeta_n$  be a primitive  $n$ -th root of unity. Set  $f_0 = \Phi_n$  and  $u = \zeta_n$ . For  $a \in \mathbb{Z}$  with  $\gcd(a, n) = 1$ , the irreducible polynomial  $f_a$  of  $\zeta_n^a$  is nothing else than  $f_0$ . Hence, if the equation

$$F_0(x, y) = \pm 1$$

has a solution  $(x, y) \in \mathbb{Z}^2$  with  $xy \neq 0$ , then for infinitely many  $a \in \mathbb{Z}$  the twisted Thue equation  $F_a(x, y) = \pm 1$  has also the solution  $(x, y)$ , since  $F_a = F_0$ . For instance, when  $n = 12$ , we have  $\Phi_{12}(X) = X^4 - X^2 + 1$  and the equation

$$x^4 - x^2y^2 + y^4 = 1$$

has the solutions  $(1, 1)$ ,  $(-1, 1)$ ,  $(1, -1)$ ,  $(-1, -1)$ .

The main result of [5], which deals only with non totally real cubic equations, is a special case of Theorem 2; the “constants” in [5] depend on  $\alpha$  and  $v$ , while here they depend only on  $d$ . The main result of [6] deals with Thue equations twisted by a set of units which is not supposed to be a group of rank 1, but it involves an assumption (namely that at least two of the conjugates of  $v$  have a modulus as large as a positive power of  $|\overline{v}|$ ) which we do not need here. Our Theorem 2 also improves the main result of [7]: we remove the assumption that the unit is totally real (besides, the result of [7] is not explicit in terms of the heights and regulator). We also notice that the part (iii) of Theorem 1.1 of [8] follows from our Theorem 2. The main result of [9] does not assume that the twists are done by a group of units of rank 1, but it needs a strong assumption which does not occur here, namely that the field  $K$  has at most one real embedding.

We conclude this §1 with some more definitions and properties.

When  $f$  is a polynomial in one variable of degree  $d$  with coefficients in  $\mathbb{Z}$  and leading coefficient  $c_0 > 0$ , the (usual) height  $H(f)$  of  $f$  is the maximum of the absolute values of the coefficients of  $f$ , while the Mahler measure of  $f$  is

$$M(f) = c_0 \prod_{i=1}^d \max\{1, |\gamma_i|\},$$

where  $\gamma_1, \gamma_2, \dots, \gamma_d$  are the roots of  $f$  in  $\mathbb{C}$ .

Let us recall<sup>1</sup> that the logarithmic height  $h(\gamma)$  of an algebraic number  $\gamma$  of degree  $d$  is  $\frac{1}{d} \log M(\gamma)$  where  $M(\gamma)$  is the Mahler measure of the irreducible polynomial of  $\gamma$ . We have

$$(4) \quad M(f) \leq \sqrt{d+1} H(f) \quad \text{and} \quad H(f) \leq 2^d M(f)$$

(see [12], Annex to Chapter 3, *Inequalities Between Different Heights of a Polynomial*, pp. 113–114; see also [2, §1.9]). The second upper bound in (4) could be replaced by the sharper one

$$H(f) \leq \binom{d}{\lfloor d/2 \rfloor} M(f),$$

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<sup>1</sup>Our  $h$  is the same as in [2], it corresponds to the logarithm of the  $h$  in [1].

but we will not need it.

Let  $v$  be a unit of degree  $d$  and conjugates  $v_1, \dots, v_d$  with

$$|v_1| \leq |v_2| \leq \dots \leq |v_d|,$$

so that  $|\overline{v}| = |v_d|$ . Let  $\lambda = M(v)$  and let  $s$  be an index in  $\{1, \dots, d-1\}$  such that

$$|v_1| \leq |v_2| \leq \dots \leq |v_s| \leq 1 \leq |v_{s+1}| \leq \dots \leq |v_d|.$$

We have

$$\lambda = M(v) = |v_{s+1} \cdots v_d| \leq |v_d|^{d-s} \leq |v_d|^{d-1}$$

and

$$M(v^{-1}) = |v_1 \cdots v_s|^{-1} = M(v) = \lambda$$

with

$$\lambda \leq |v_1|^{-s} \leq |v_1|^{-(d-1)}.$$

Therefore we have

$$(5) \quad \lambda^{1/(d-1)} \leq |v_d| \leq \lambda \quad \text{and} \quad \lambda^{-1} \leq |v_1| \leq \lambda^{-1/(d-1)}.$$

## 2 Examples

The lower bound  $\mu \geq \lambda^{\kappa_4}$  quoted in section 1 is true

- when  $d = 3$  and the cubic field  $K$  is not totally real;
- for the simplest fields of degree 3 (see [8]), and also for the simplest fields of degrees 4 and 6;
- when  $-v$  is a Galois conjugate of  $v$  (which means that the irreducible polynomial of  $v$  is in  $\mathbb{Z}[X^2]$ ), and more generally when  $|v_1| = |v_2|$  and  $|v_{d-1}| = |v_d|$  with  $d \geq 4$ .

Here is an example of this last situation. Let  $\epsilon$  be an algebraic unit, not a root of unity, of degree  $\ell \geq 2$  and conjugates  $\epsilon_1, \epsilon_2, \dots, \epsilon_\ell$ . Let  $h \geq 2$  and let  $d = \ell h$ . For  $a \in \mathbb{Z}$ , define

$$(6) \quad F_a(X, Y) = \prod_{i=1}^{\ell} (X^h - \epsilon_i^a Y^h).$$

Let  $R$  be the regulator of the field  $\mathbb{Q}(\epsilon^{1/h})$ .

From Theorem 2 we deduce the following corollary.

**Corollary 3.** *Let  $m \geq 1$ . If the form  $F_a$  in (6) is irreducible and if there exists  $(x, y) \in \mathbb{Z}^2$  with  $xy \neq 0$  and  $|F_a(x, y)| \leq m$ , then*

$$|a| \leq \kappa_6(R + \log m + \log |\epsilon|) R \log^*(R \log |\epsilon|).$$

PROOF. Without loss of generality, assume  $|\epsilon_1| \leq |\epsilon_2| \leq \dots \leq |\epsilon_\ell|$ , so that  $|\epsilon_\ell| = \lceil \epsilon \rceil$ . Let  $\zeta$  be a primitive  $h$ -th root of unity. Let  $v = \epsilon^{1/h}$ . We apply Theorem 2 with  $\alpha = \zeta$ ,  $a_0 = 1$ ,  $\lambda_0 = 1$ ,  $\lambda \leq \lceil \epsilon \rceil^\ell$ ,  $F_0(X, Y) = (X^h - Y^h)^\ell$  and

$$v_{ih+j} = \zeta^{j-1} \epsilon_{i+1}^{1/h} \quad (0 \leq i \leq \ell - 1, 1 \leq j \leq h).$$

From  $|v_1| = |v_2| = |\epsilon_1|^{1/h} < 1$  and  $|v_{d-1}| = |v_d| = |\epsilon_\ell|^{1/h}$  we deduce

$$\mu = \left| \frac{\epsilon_\ell}{\epsilon_1} \right|^{1/h} = \left| \frac{v_d}{v_1} \right|$$

and using (5) we conclude

$$\log \mu \geq \frac{2}{d-1} \log \lambda. \quad \square$$

A variant of this proof is to take  $\alpha = 1$ ,  $\lambda_0 = 1$ ,  $F_0(X, Y) = (X - Y)^d$ , and to use the fact that  $\zeta^a$  is also a primitive  $h$ -th root of unity since  $F_a$  is irreducible.

### 3 Auxiliary results

#### 3.1 An elementary result

For the convenience of the reader, we include the following elementary result – similar arguments are often used without explicit mention in the literature.

**Lemma 4.** *Let  $U$  and  $V$  be positive numbers satisfying  $U \leq V \log^* U$ . Then  $U < 2V \log^* V$ .*

**Proof.** If  $\log U \leq 1$ , the assumption is  $U \leq V$  and the conclusion follows. Assume  $\log U > 1$ . Then  $\log U \leq \sqrt{\log U}$ , hence the hypothesis of the lemma implies  $U \leq V \sqrt{\log U}$  and therefore we have  $U \leq V^2$ . We deduce

$$\log U \leq 2 \log V,$$

hence

$$U \leq V \log U \leq 2V \log V. \quad \square$$

#### 3.2 Diophantine tool

In this section only, the positive integer  $d$  is not restricted to  $d \geq 3$ .

The main tool is the following Diophantine estimate ([6, Proposition 2], [12, Theorem 9.1] or [2, Th. 3.2.4]), the proof of which uses transcendental number theory.

**Proposition 5.** *Let  $s$  and  $D$  be two positive integers. There exists an effectively computable positive constant  $\kappa(s, D)$ , depending only upon  $s$  and  $D$ , with the following property. Let  $\eta_1, \dots, \eta_s$  be nonzero algebraic numbers generating a number field of degree  $\leq D$ . Let  $c_1, \dots, c_s$  be rational integers and let  $H_1, \dots, H_s$  be real numbers  $\geq 1$  satisfying*

$$H_i \geq h(\eta_i) \quad (1 \leq i \leq s).$$

*Let  $C$  be a real number with  $C \geq 2$ . Suppose that one of the following two statements is true:*

(i)  $C \geq \max_{1 \leq j \leq s} |c_j|$

*or*

(ii)  $H_j \leq H_s$  for  $1 \leq j \leq s$  and

$$C \geq \max_{1 \leq j \leq s} \left\{ \frac{H_j}{H_s} |c_j| \right\}.$$

*Suppose also  $\eta_1^{c_1} \cdots \eta_s^{c_s} \neq 1$ . Then*

$$|\eta_1^{c_1} \cdots \eta_s^{c_s} - 1| > \exp\{-\kappa(s, D)H_1 \cdots H_s \log C\}.$$

The statement (ii) of Proposition 5 implies the statement (i) by permuting the indices so that  $H_j \leq H_s$  for  $1 \leq j \leq s$ ; however, we find it more convenient to use the part (i) so that we can use the estimate without permuting the indices.

We will use Proposition 5 several times. Here is a first consequence.

**Corollary 6.** *Let  $d \geq 1$ . There exists a constant  $\kappa_7$ , which depends only on  $d$ , with the following property. Let  $K$  be a number field of degree  $d$ . Let  $\alpha_1, \alpha_2, v_1, v_2$  be nonzero elements in  $K$  and let  $a$  be a nonzero integer. Set  $\gamma_1 = \alpha_1 v_1^a$  and  $\gamma_2 = \alpha_2 v_2^a$ . Let  $\lambda_0$  and  $\lambda$  satisfy*

$$\max\{h(\alpha_1), h(\alpha_2)\} \leq \log \lambda_0, \quad \max\{h(v_1), h(v_2)\} \leq \log \lambda$$

*and assume  $\gamma_1 \neq \gamma_2$ . Define*

$$\chi = (\log^* \lambda_0)(\log^* \lambda) \log^* \left( |a| \min \left\{ 1, \frac{\log^* \lambda}{\log^* \lambda_0} \right\} \right).$$

*Then*

$$|\gamma_1 - \gamma_2| \geq \max\{|\gamma_1|, |\gamma_2|\} e^{-\kappa_7 \chi}.$$

**Proof.** By symmetry, without loss of generality, we may assume  $|\gamma_2| \geq |\gamma_1|$ . Set

$$s = 2, \quad \eta_1 = \frac{v_1}{v_2}, \quad \eta_2 = \frac{\alpha_1}{\alpha_2}, \quad c_1 = a, \quad c_2 = 1,$$

$$H_1 = 2 \log^* \lambda, \quad H_2 = 2 \log^* \lambda_0, \quad C = \max \left\{ 2, |a| \min \left\{ 1, \frac{H_1}{H_2} \right\} \right\}.$$

The conclusion of Corollary 6 follows from Proposition 5 (via part (i) if  $H_1 \geq H_2$ , via part (ii) otherwise), thanks to the relation

$$|\eta_1^{c_1} \eta_2^{c_2} - 1| = |\gamma_2|^{-1} |\gamma_1 - \gamma_2|. \quad \square$$

### 3.3 Lower bound for the height and the regulator

For the record, we quote Kronecker's Theorem and its effective improvement.

**Lemma 7.** (a) *If a nonzero algebraic integer  $\alpha$  has all its conjugates in the closed unit disc  $\{z \in \mathbb{C} \mid |z| \leq 1\}$ , then  $\alpha$  is a root of unity.*

(b) *More precisely, given  $d \geq 1$ , there exists an effectively computable positive constant  $\kappa_8$ , depending only on  $d$ , such that, if  $\alpha$  is a nonzero algebraic integer of degree  $d$  satisfying  $h(\alpha) < \kappa_8$ , then  $\alpha$  is a root of unity.*

**Proof.** Voutier (1996) refined an earlier estimate due to Dobrowolski (1979) by proving that the conclusion of the part (b) in Lemma 7 holds with

$$\kappa_8 = \begin{cases} \log 2 & \text{if } d = 1, \\ 2d(\log d)^3 & \text{if } d \geq 2. \end{cases}$$

See for instance [2, Prop. 3.2.9] and [12, §3.6].  $\square$

**Lemma 8.** *There exists an explicit absolute constant  $\kappa_9 > 0$  such that the regulator  $R$  of any number field of degree  $\geq 2$  satisfies  $R > \kappa_9$ .*

**Proof.** According to a result of Friedman (1989 – see [2, (1.5.3)]) the conclusion of Lemma 8 holds with  $\kappa_9 = 0.2052$ .  $\square$

### 3.4 A basis of units of an algebraic number field

Here is Lemma 1 of [1]. See also [2, Proposition 4.3.9]. The result is essentially due to C.L. Siegel [11].

**Proposition 9.** *Let  $d$  be a positive integer with  $d \geq 3$ . There exist effectively computable constants  $\kappa_{10}, \kappa_{11}, \kappa_{12}$  depending only on  $d$ , with the following property. Let  $K$  be a number field of degree  $d$ , with unit group of rank  $r$ . Let  $R$  be the regulator of this field. Denote by  $\varphi_1, \varphi_2, \dots, \varphi_r$  a set of  $r$  embeddings of  $K$  into  $\mathbb{C}$  containing the real embeddings and no pair of conjugate embeddings. Then there exists a fundamental system of units  $\{\epsilon_1, \epsilon_2, \dots, \epsilon_r\}$  of  $K$  which satisfies the following:*

$$(i) \quad \prod_{1 \leq i \leq r} h(\epsilon_i) \leq \kappa_{10} R;$$

$$(ii) \quad \max_{1 \leq i \leq r} h(\epsilon_i) \leq \kappa_{11} R;$$

(iii) *The absolute values of the entries of the inverse matrix of*

$$(\log |\varphi_j(\epsilon_i)|)_{1 \leq i, j \leq r}$$

*do not exceed  $\kappa_{12}$ .*



The next result is [10, Lemma A.15].

**Lemma 10.** *Let  $\epsilon_1, \epsilon_2, \dots, \epsilon_r$  be an independent system of units for  $K$  satisfying the condition (ii) of Proposition 9. Let  $\beta \in \mathbb{Z}_K$  with  $N_{K/\mathbb{Q}}(\beta) = m \neq 0$ . Then there exist  $b_1, b_2, \dots, b_r$  in  $\mathbb{Z}$  and  $\tilde{\beta} \in \mathbb{Z}_K$  with conjugates  $\tilde{\beta}_1, \tilde{\beta}_2, \dots, \tilde{\beta}_d$ , satisfying*

$$\beta = \tilde{\beta} \epsilon_1^{b_1} \epsilon_2^{b_2} \cdots \epsilon_r^{b_r}$$

and

$$|m|^{1/d} e^{-\kappa_{13} R} \leq |\tilde{\beta}_j| \leq |m|^{1/d} e^{\kappa_{13} R} \quad \text{for } j = 1, \dots, d.$$

The conclusion of Lemma 10 can be written

$$\left| \log \left( |m|^{-1/d} |\tilde{\beta}_j| \right) \right| \leq \kappa_{13} R \quad \text{for } j = 1, \dots, d.$$

### 3.5 Estimates for the conjugates

**Lemma 11.** *Let  $\gamma$  be an algebraic number of degree  $d \geq 3$ . Let  $\gamma_1, \gamma_2, \dots, \gamma_d$  be the conjugates of  $\gamma$  with  $|\gamma_1| \leq |\gamma_2| \leq \dots \leq |\gamma_d|$ .*

- (a) *If  $|\gamma_1| < |\gamma_2|$  and  $\gamma_2 \in \mathbb{R}$ , then  $|\gamma_2| < |\gamma_3|$ .*
- (b) *If  $|\gamma_{d-1}| < |\gamma_d|$  and  $\gamma_{d-1} \in \mathbb{R}$ , then  $|\gamma_{d-2}| < |\gamma_{d-1}|$ .*

**Proof.** (a) The conditions  $|\gamma_1| < |\gamma_2| \leq |\gamma_i|$  for  $3 \leq i \leq d$  imply that  $\gamma_1$  is real and that  $-\gamma_1$  is not a conjugate of  $\gamma_1$ . Hence the minimal polynomial of  $\gamma$  is not a polynomial in  $X^2$ . Assume  $|\gamma_2| = |\gamma_3|$ . Since  $-\gamma_2$  is not a conjugate of  $\gamma_2$ , we deduce  $\gamma_3 \notin \mathbb{R}$ , hence  $d \geq 4$ . We may assume  $\gamma_4 = \overline{\gamma_3}$ . Let  $\sigma$  be an automorphism of  $\overline{\mathbb{Q}}$  which maps  $\gamma_2$  to  $\gamma_1$ ; via  $\sigma$ , let  $\gamma_j$  be the image of  $\gamma_3$  and  $\gamma_k$  the image of  $\gamma_4$ . From

$$\gamma_2^2 = \gamma_3 \gamma_4$$

we deduce  $\gamma_1^2 = \gamma_j \gamma_k$  and  $|\gamma_1|^2 = |\gamma_j \gamma_k|$ . This is not possible since  $|\gamma_j| > |\gamma_1|$  and  $|\gamma_k| > |\gamma_1|$ .

(b) We deduce (b) from (a), by using  $\gamma \mapsto 1/\gamma$  (or by repeating the proof, *mutatis mutandis*).  $\square$

**Remark.** Here is an example showing that the assumptions of Lemma 11 are sharp. The polynomial  $X^4 - 4X^2 + 1$  is irreducible, its roots are

$$v_1 = \sqrt{2 - \sqrt{3}}, \quad v_2 = -v_1, \quad v_3 = 1/v_1 = \sqrt{2 + \sqrt{3}}, \quad v_4 = -v_3$$

with

$$v_1 = |v_2| < v_3 = |v_4|.$$

More generally, if  $h \geq 2$  is a positive integer and  $\epsilon$  is a quadratic unit with Galois conjugate  $\epsilon'$  and if  $\epsilon^{1/h}$  has degree  $2h$ , then it has  $h$  conjugates of absolute value  $|\epsilon|^{1/h}$  and  $h$  conjugates of absolute value  $|\epsilon'|^{1/h}$ . See also §2.

**Lemma 12.** *Let  $v$  be an algebraic unit of degree  $d \geq 3$ . Set  $\lambda = M(v)$ . Let  $v'$  and  $v''$  be two conjugates of  $v$  with  $|v'| < |v''|$ . Then*

$$\log \frac{|v''|}{|v'|} \geq \kappa_{14} \lambda^{-(d^3+2d^2-d+2)/2}.$$

We will deduce Lemma 12 from Theorem 1 of [3] which<sup>2</sup> states the following.

**Lemma 13** (X. Gourdon and B. Salvy [3]). *Let  $P$  be a polynomial of degree  $d \geq 2$  with integer coefficients and with Mahler measure  $M(P)$ . If  $\alpha'$  and  $\alpha''$  are two roots of  $P$  with  $|\alpha'| < |\alpha''|$ , then*

$$|\alpha''| - |\alpha'| \geq \kappa_{15} M(P)^{-d(d^2+2d-1)/2}$$

with

$$\kappa_{15} = \frac{\sqrt{3}}{2} (d(d+1)/2)^{-d(d+1)/4-1}.$$

**Proof of Lemma 12.** We apply Lemma 13 to the minimal polynomial of  $v$ . To conclude the proof of Lemma 12, we use the bounds  $|v'| \leq \lambda$  and

$$\log(1+x) \geq \frac{x}{2} \quad \text{for } 0 \leq x \leq 1 \quad \text{with } x = \frac{|v''|}{|v'|} - 1. \quad \square$$

## 4 Proof of Theorem 2

Theorem 2 with the assumption  $|F_a(x, y)| \leq m$  will be secured if we deal with the equation  $F_a(x, y) = m$  with  $m \neq 0$ .

Let  $(a, x, y, m) \in \mathbb{Z}^4$  satisfy  $m \neq 0$ ,  $xy \neq 0$ ,  $[\mathbb{Q}(\alpha v^a) : \mathbb{Q}] = d$  and

$$F_a(x, y) = m.$$

Without loss of generality, we may restrict  $(a, y)$  to  $a \geq 0$  (otherwise, replace  $v$  by  $v^{-1}$ ) and to  $y > 0$  (otherwise replace  $F_a(X, Y)$  by  $F_a(X, -Y)$ ).

The form  $\tilde{F}_a(X, Y) = a_0^{d-1} F_a(X, Y)$  has coefficients in  $\mathbb{Z}$ , and if we set  $\tilde{x} = a_0 x$ ,  $\tilde{y} = y$ ,  $\tilde{m} = a_0^{d-1} m$  we have  $\tilde{F}_a(\tilde{x}, \tilde{y}) = \tilde{m}$  with  $(\tilde{x}, \tilde{y}) \in \mathbb{Z}^2$ . Therefore, there is no loss of generality to assume  $a_0 = 1$ .

Theorem 2 includes the assumption that  $v$  is not a root of unity, hence  $\lambda > 1$ . More precisely, it follows from the part (b) of Lemma 7 that

$$\log \lambda \geq \kappa_8.$$

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<sup>2</sup>This reference was kindly suggested to us by Yann Bugeaud.

In particular, we have

$$\log^* \lambda \leq \max \left\{ 1, \frac{1}{\kappa_8} \right\} \log \lambda,$$

an inequality which can be written

$$(7) \quad \log^* \lambda \leq \kappa_{16} \log \lambda$$

with an effectively computable constant  $\kappa_{16} > 0$ .

From Lemma 8, we deduce that  $R > \kappa_9$ . Therefore, there is no loss of generality to assume that, for a sufficiently large constant  $\kappa_{17}$ , we have

$$(8) \quad a \geq \kappa_{17} (\log |m| + (\log^* \lambda_0) \log^* \log^* \lambda).$$

This hypothesis will frequently be used, sometimes without explicit mention.

By assumption,  $\mathbb{Q}(\alpha v^a) = K$ . If some conjugate  $\sigma_j(\alpha v^a)$  of  $\alpha v^a$  is real, then it follows that  $\sigma_j(K) \subset \mathbb{R}$ , hence the embedding  $\sigma_j$  is real, and  $\alpha_j$  and  $v_j$  are both real. We also notice that if  $\sigma_j(v) = -\sigma_i(v)$  with  $i \neq j$ , then it follows that  $v$  and  $-v$  are conjugate, hence the irreducible polynomial of  $v$  belongs to  $\mathbb{Z}[X^2]$ .

Recall that  $v_i = \sigma_i(v)$  ( $i = 1, \dots, d$ ) and that

$$|v_1| \leq |v_2| \leq \dots \leq |v_d|.$$

Let us write  $\alpha_i$  for  $\sigma_i(\alpha)$  ( $i = 1, \dots, d$ ). Let

$$\gamma = \alpha v^a \quad \text{and} \quad \beta = x - \gamma y.$$

Since  $a_0 = 1$ , it follows that  $\alpha$ ,  $\beta$  and  $\gamma$  are algebraic integers in  $K$ . For  $j = 1, 2, \dots, d$ , define  $\gamma_j$  and  $\beta_j$  by

$$\gamma_j = \sigma_j(\gamma) = \alpha_j v_j^a, \quad \beta_j = \sigma_j(\beta) = x - \alpha_j v_j^a y = x - \gamma_j y.$$

The assumption  $F_a(x, y) = m$  yields  $\beta_1 \beta_2 \dots \beta_d = m$ . Let  $i_0 \in \{1, 2, \dots, d\}$  be an index such that

$$|\beta_{i_0}| = \min_{1 \leq i \leq d} |\beta_i|.$$

We define  $\Psi_1, \Psi_2, \dots, \Psi_d$  by the following conditions:

$$\beta_i = \begin{cases} \gamma_{i_0} y \Psi_i & \text{for } 1 \leq i < i_0, \\ \gamma_i y \Psi_i & \text{for } i_0 < i \leq d \end{cases}$$

and

$$\beta_{i_0} = \frac{m}{y^{d-1}} \cdot \frac{\gamma_1 \gamma_2 \dots \gamma_{i_0-1}}{\gamma_{i_0}^{i_0-2}} \Psi_{i_0}.$$

We split the proof into several steps.

**Step 1.** We start by proving that

$$(9) \quad |x| \leq 2\lambda_0\lambda^ay$$

and that there exists an effectively computable positive constant  $\kappa_{18}$  depending only on  $d$  such that

$$(10) \quad e^{-\kappa_{18}\chi} \leq |\Psi_i| \leq e^{\kappa_{18}\chi} \quad (i = 1, 2, \dots, d)$$

with

$$\chi = (\log^* \lambda_0)(\log \lambda) \log \left( a \min \left\{ 1, \frac{\log \lambda}{\log^* \lambda_0} \right\} \right).$$

From the estimate (10) we will deduce

$$|\beta_{i_0}| < |\beta_i|$$

for  $i \neq i_0$ , which implies  $\alpha_{i_0} \in \mathbb{R}$  and  $v_{i_0} \in \mathbb{R}$ .

**Remark.** The estimate (10) can be written as follows:

$$\left| \log (|\beta_i|y^{-1} |\max\{|\gamma_i^{-1}|, |\gamma_{i_0}^{-1}|\})| \right| \leq \kappa_{18}\chi$$

for  $i \neq i_0$  and

$$\left| \log \left( |\beta_{i_0}| \frac{y^{d-1}}{|m|} \left| \gamma_1^{-1} \cdots \gamma_{i_0-1}^{-1} \gamma_{i_0}^{i_0-2} \right| \right) \right| \leq \kappa_{18}\chi.$$

**Proof** of (9) and (10). We have

$$(11) \quad |x| = |\beta_{i_0} + \gamma_{i_0}y| \leq |\beta_{i_0}| + |\gamma_{i_0}|y.$$

From  $|\beta_{i_0}| \leq |\beta_i|$  for  $i = 1, 2, \dots, d$  and  $\beta_1 \cdots \beta_d = m$ , we deduce  $|\beta_{i_0}| \leq |m|^{1/d}$ , hence

$$|x| \leq |m|^{1/d} + |\gamma_{i_0}|y \leq |m|^{1/d} + \lambda_0\lambda^ay.$$

Using the assumption (8), we check  $|m|^{1/d} \leq \lambda_0\lambda^ay$ , whereupon the inequality (9) is secured.

We also have

$$(12) \quad |\beta_{i_0}|^{d-1} \max_{1 \leq i \leq d} |\beta_i| \leq |m|.$$

For  $i = 1, 2, \dots, d$ , we write

$$(13) \quad \beta_i = \beta_{i_0} + y(\gamma_{i_0} - \gamma_i).$$

We have

$$|\alpha_1\alpha_2 \cdots \alpha_d| \geq 1$$

(recall  $a_0 = 1$ ), hence

$$\frac{1}{\lambda_0} \leq |\alpha_i| \leq \lambda_0 \quad \text{for } i = 1, 2, \dots, d.$$

We choose an index  $j_0 \neq i_0$  as follows:

- If  $|v_{i_0}| \leq \lambda^{1/(2(d-1))}$ , we take  $j_0 = d$  so that, with the help of (5), we have  $|v_{j_0}| \geq \lambda^{1/(d-1)}$ , whereupon with the help of (8) we obtain

$$\left| \frac{\gamma_{i_0}}{\gamma_{j_0}} \right| < \frac{1}{2}.$$

- If  $|v_{i_0}| > \lambda^{1/(2(d-1))}$ , we take  $j_0 = 1$  so that, again with the help of (5), we have  $|v_{j_0}| \leq \lambda^{-1/(d-1)}$ , whereupon with the help of (8) we obtain

$$\left| \frac{\gamma_{j_0}}{\gamma_{i_0}} \right| < \frac{1}{2}.$$

In both cases, we deduce

$$|\gamma_{j_0} - \gamma_{i_0}| \geq \frac{1}{2} \max\{|\gamma_{j_0}|, |\gamma_{i_0}|\} \geq \frac{\lambda^{a/(2(d-1))}}{2\lambda_0}$$

and therefore, using (8) again together with (12) and (13), we obtain

$$|\beta_{j_0}| \geq |\gamma_{j_0} - \gamma_{i_0}|y - |\beta_{i_0}| \geq \frac{\lambda^{a/(2(d-1))}y}{2\lambda_0} - |m|^{1/d} \geq \lambda^{a/(2d)}y.$$

Since  $\max_{1 \leq i \leq d} |\beta_i| \geq \lambda^{a/(2d)}y$ , from (12) we deduce

$$(14) \quad |\beta_{i_0}| \leq \left( \frac{|m|}{y\lambda^{a/(2d)}} \right)^{1/(d-1)}.$$

In particular, thanks to (8), we have

$$(15) \quad |\beta_{i_0}| \leq \frac{1}{2}.$$

Using the assumption  $|x| \geq 1$  together with (11), we deduce

$$(16) \quad \frac{|x|}{2} \leq |\gamma_{i_0}|y \leq |x| + |\beta_{i_0}| \leq \frac{3|x|}{2}.$$

Let  $i \neq i_0$ . The upper bound

$$|\gamma_i - \gamma_{i_0}| \leq 2 \max\{|\gamma_{i_0}|, |\gamma_i|\}$$

is trivial, while the lower bound

$$(17) \quad |\gamma_i - \gamma_{i_0}| \geq \max\{|\gamma_{i_0}|, |\gamma_i|\} e^{-\kappa_{19}X}$$

follows from (7) and from Corollary 6. We first use the lower bound

$$|\gamma_i - \gamma_{i_0}| \geq |\gamma_{i_0}| e^{-\kappa_{19}\chi}.$$

Using (16), we obtain

$$(18) \quad |\gamma_i - \gamma_{i_0}| \geq \frac{1}{2y} e^{-\kappa_{19}\chi} \geq \frac{2}{y} e^{-\kappa_{20}\chi}$$

with  $\kappa_{20} > 0$ . Using the contrapositive of Lemma 4 with

$$U = a \frac{\log^* \lambda}{\log^* \lambda_0}, \quad V = \frac{1}{\kappa_{21}} \log^* \lambda,$$

we deduce from (8) that

$$\chi \leq \kappa_{21} a \log^* \lambda.$$

Recall that  $\kappa_{17}$  is sufficiently large, hence  $\kappa_{21}$  is sufficiently small. Now from (14), the inequality  $|m| \leq e^{a/\kappa_{17}}$  and (18) we deduce

$$|\beta_{i_0}| \leq |m|^{1/(d-1)} \lambda^{-a/(2d(d-1))} \leq \lambda^{-\kappa_{22}a} \leq e^{-\kappa_{20}\chi} \leq \frac{1}{2} y |\gamma_i - \gamma_{i_0}|.$$

Therefore, for  $i \neq i_0$ , using (13), we deduce

$$\frac{1}{2} y |\gamma_i - \gamma_{i_0}| \leq |\beta_i| \leq \frac{3}{2} y |\gamma_i - \gamma_{i_0}|.$$

Using once more (17), we obtain (10) for  $i \neq i_0$ . We also deduce

$$(19) \quad |\beta_i| > \lambda^{-a/(2d)} \quad \text{for } i \neq i_0.$$

Recall

$$N(\gamma) = \gamma_1 \gamma_2 \cdots \gamma_d = N(\alpha) N(v)^a = \pm N(\alpha) \quad \text{and} \quad N(\beta) = \beta_1 \beta_2 \cdots \beta_d = m.$$

The estimate (10) for  $i = i_0$  follows from the relations

$$\Psi_1 \Psi_2 \cdots \Psi_d N(\gamma) = 1,$$

$$\frac{m}{\beta_{i_0}} = \prod_{i \neq i_0} \beta_i = y^{d-1} \gamma_{i_0}^{i_0-1} \gamma_{i_0+1} \cdots \gamma_d \prod_{i \neq i_0} \Psi_i$$

and

$$\frac{N(\gamma)}{\gamma_{i_0}^{i_0-1} \gamma_{i_0+1} \cdots \gamma_d} = \frac{\gamma_1 \cdots \gamma_{i_0-1}}{\gamma_{i_0}^{i_0-2}}.$$

From (8) and (10), we deduce

$$|\beta_{i_0}| < \frac{|m|}{y^{d-1}} |\gamma_1| e^{\kappa_{18}\chi} < \lambda^{-a/(2d)},$$

hence from (19) we infer  $|\beta_{i_0}| < |\beta_i|$  for  $i \neq i_0$ . It follows that  $\beta_{i_0}$  is real, and therefore  $\gamma_{i_0}$ ,  $\alpha_{i_0}$  and  $v_{i_0}$  also.  $\square$

**Step 2.** Let  $\epsilon_1, \epsilon_2, \dots, \epsilon_r$  be a basis of the group of units of  $K$  given by Proposition 9. From Lemma 10, it follows that there exists  $\tilde{\beta} \in \mathbb{Z}_K$  and  $b_1, b_2, \dots, b_r$  in  $\mathbb{Z}$  with

$$\beta = \tilde{\beta} \epsilon_1^{b_1} \epsilon_2^{b_2} \dots \epsilon_r^{b_r}$$

and

$$|m|^{1/d} e^{-\kappa_{13} R} \leq |\tilde{\beta}_i| \leq |m|^{1/d} e^{\kappa_{13} R} \quad \text{for } i = 1, 2, \dots, d.$$

We set

$$(20) \quad B = \kappa_{23}(R + a \log \lambda + \log y)$$

with a sufficiently large constant  $\kappa_{23}$ . We want to prove that

$$\max_{1 \leq i \leq r} |b_i| \leq B.$$

**Proof.** We consider the system of  $d$  linear forms in  $r$  variables with real coefficients

$$L_i(X_1, X_2, \dots, X_r) = \sum_{j=1}^r X_j \log |\sigma_i(\epsilon_j)|, \quad (i = 1, 2, \dots, d).$$

The rank is  $r$ . By Proposition 9(ii),

$$\log |\sigma_i(\epsilon_j)| \leq \kappa_{24} R.$$

For  $i = 1, 2, \dots, d$ , define  $e_i = L_i(b_1, b_2, \dots, b_r)$ . We have

$$e_i = \log |\sigma_i(\beta/\tilde{\beta})| = \log |\beta_i/\tilde{\beta}_i|,$$

hence, using the inequality  $|m| \leq e^{|a|/\kappa_{17}}$  and (10), we deduce

$$|e_i| \leq \kappa_{25}(R + a \log \lambda + \log y).$$

Computing  $b_1, b_2, \dots, b_r$  by means of the system of linear equations

$$L_i(b_1, b_2, \dots, b_r) = e_i \quad (i = 1, 2, \dots, d)$$

and using Proposition 9(iii), we deduce

$$\max_{1 \leq j \leq r} |b_j| \leq \kappa_{26} \max_{1 \leq i \leq d} |e_i| \leq B. \quad \square$$

**Step 3.** From the inequality (3.2) in [1, Theorem 3] (see also [2, Th. 9.6.2]), thanks to (8), we deduce the following upper bound for  $|x|$  and  $|y|$  in terms

of  $a, \lambda, \lambda_0, m$  and  $R$ : there exists a positive effectively computable constant  $\kappa_{27}$  depending only on  $d$  such that

$$(21) \quad \log \max\{|x|, y\} \leq \kappa_{27} R (\log^* R) (R + a \log \lambda).$$

**Step 4.** Assume  $c\gamma_i\beta_j \neq \gamma_k\beta_\ell$  for some indices  $i, j, k, \ell$  in  $\{1, \dots, d\}$  and some  $c \in \{1, -1\}$ . Then there exists  $\kappa_{28} > 0$  such that

$$\left| c \frac{\gamma_i\beta_j}{\gamma_k\beta_\ell} - 1 \right| \geq \exp \left\{ -\kappa_{28} (\log \lambda) (R + \log |m| + \log \lambda_0 + \log \lambda) R \right. \\ \left. \times \log \left( \frac{Ra \log \lambda}{R + \log |m| + \log \lambda_0 + \log \lambda} \right) \right\}.$$

**Proof.** This lower bound follows from Proposition 5(ii) with

$$\frac{c\gamma_i\beta_j}{\gamma_k\beta_\ell} = \eta_1^{c_1} \eta_2^{c_2} \cdots \eta_s^{c_s},$$

where  $s = r + 2$  and

$$\eta_t = \frac{\sigma_j(\epsilon_t)}{\sigma_\ell(\epsilon_t)} \quad (1 \leq t \leq r), \quad \eta_{r+1} = \frac{\sigma_i(v)}{\sigma_k(v)}, \quad \eta_{r+2} = \frac{c\sigma_j(\tilde{\beta})\sigma_i(\alpha)}{\sigma_\ell(\tilde{\beta})\sigma_k(\alpha)},$$

$$c_t = b_t \quad (1 \leq t \leq r), \quad c_{r+1} = a, \quad c_{r+2} = 1,$$

$$H_t = \max\{1, 2h(\epsilon_t)\} \quad (1 \leq t \leq r),$$

$$H_{r+1} = \max\{1, 2 \log \lambda\}, \quad H_{r+2} = \kappa_{29} (R + \log |m| + \log \lambda_0 + \log \lambda),$$

$$C = 2 + \frac{2a \log \lambda + RB}{H_{r+2}}.$$

Using Proposition 9(i) together with the part (b) of Lemma 7, we deduce

$$H_1 H_2 \cdots H_r \leq \kappa_{30} R.$$

Finally we deduce from the steps 2 and 3 that

$$\log C \leq \kappa_{31} \log \left( \frac{Ra \log \lambda}{R + \log |m| + \log \lambda_0 + \log \lambda} \right),$$

and this secures the above linear bound. □

**Step 5.** We will prove Theorem 2 by assuming

$$\max_{1 \leq i \leq d} \max\{|\Psi_i|, |\Psi_i|^{-1}\} > \mu^{a/4}.$$

Using (10), we deduce from our assumption

$$\frac{a}{4} \log \mu < \kappa_{18} \chi,$$



hence

$$a \leq \frac{4\kappa_{18}}{\log \mu} (\log^* \lambda_0) (\log^* \lambda) \log^* \left( a \frac{\log^* \lambda}{\log^* \lambda_0} \right).$$

With

$$U = \frac{a \log^* \lambda}{\log^* \lambda_0} \quad \text{and} \quad V = \frac{4\kappa_{18} (\log^* \lambda)^2}{\log \mu},$$

we have  $U \leq V \log^* U$ , and we conclude that we can use Lemma 4 to deduce

$$a \leq \frac{8\kappa_{18} (\log^* \lambda_0) (\log^* \lambda)}{\log \mu} \log \left( \frac{4\kappa_{18} (\log^* \lambda)^2}{\log \mu} \right),$$

and the conclusion of Theorem 2 follows.

In the rest of the paper, we assume

$$(22) \quad \max_{1 \leq i \leq d} \max\{|\Psi_i|, |\Psi_i|^{-1}\} \leq \mu^{a/4}.$$

**Step 6.** Our next goal is to prove the following results.

(a) Assume  $1 \leq i_0 \leq d-2$  and

$$\frac{|v_{d-1}|}{|v_{i_0}|} \geq \sqrt{\mu}.$$

Then

$$0 < \left| \frac{\gamma_{d-1}\beta_d}{\gamma_d\beta_{d-1}} - 1 \right| \leq 4\lambda_0^2 \mu^{-a/4}.$$

(b) Assume  $3 \leq i_0 \leq d$  and

$$\frac{|v_{i_0}|}{|v_2|} \geq \sqrt{\mu}.$$

Then

$$0 < \left| \frac{\beta_1}{\beta_2} - 1 \right| \leq 2\lambda_0^2 \mu^{-a/4}.$$

(c) Assume  $2 \leq i_0 \leq d-1$  and

$$\min \left\{ \frac{|v_{i_0}|}{|v_1|}, \frac{|v_d|}{|v_{i_0}|} \right\} \geq \mu.$$

Then

$$\left| \frac{\gamma_{i_0}\beta_d}{\gamma_d\beta_1} + 1 \right| \leq 4|m|\lambda_0^4 \mu^{-a/2}.$$

**Proof** (a) We approximate  $\beta_d$  by  $-\gamma_d y$ ,  $\beta_{d-1}$  by  $-\gamma_{d-1} y$  and we eliminate  $y$ . Since  $\gamma$  has degree  $d$ , we have

$$\beta_d \gamma_{d-1} - \beta_{d-1} \gamma_d = x(\gamma_{d-1} - \gamma_d) \neq 0.$$

From (16) we deduce  $|x| \leq 2|\gamma_{i_0}y|$  and

$$|\beta_d\gamma_{d-1} - \beta_{d-1}\gamma_d| \leq 2|\gamma_{i_0}|(|\gamma_d| + |\gamma_{d-1}|)y.$$

Using  $\beta_{d-1} = \gamma_{d-1}y\Psi_{d-1}$  together with the assumption

$$|v_d| \geq |v_{d-1}| \geq \sqrt{\mu}|v_{i_0}|,$$

we deduce

$$\left| \frac{\gamma_{d-1}\beta_d}{\gamma_d\beta_{d-1}} - 1 \right| \leq \frac{2|\gamma_{i_0}|(|\gamma_{d-1}| + |\gamma_d|)}{|\gamma_{d-1}\gamma_d|} |\Psi_{d-1}|^{-1} \leq 4\lambda_0^2\mu^{-a/2} |\Psi_{d-1}|^{-1}.$$

The conclusion of (a) follows from (22).

(b) We approximate  $\beta_1$  and  $\beta_2$  by  $x$  and we eliminate  $x$ . Since  $\gamma_1 \neq \gamma_2$ , we have

$$|\beta_1 - \beta_2| = |(\gamma_2 - \gamma_1)y| \neq 0.$$

From  $\beta_2 = \gamma_{i_0}y\Psi_2$  and the assumption

$$|v_1| \leq |v_2| \leq \mu^{-1/2}|v_{i_0}|,$$

we deduce

$$\left| \frac{\beta_1}{\beta_2} - 1 \right| \leq \frac{|\gamma_2| + |\gamma_1|}{|\gamma_{i_0}|} |\Psi_2|^{-1} \leq 2\lambda_0^2\mu^{-a/2} |\Psi_2|^{-1}.$$

Again, the conclusion of (b) follows from (22).

(c) We approximate  $\beta_1$  by  $x$ ,  $\beta_d$  by  $-y\gamma_d$  and  $x$  by  $y\gamma_{i_0}$ , then we eliminate  $x$  and  $y$ . More precisely we have

$$\beta_1\gamma_d + \beta_d\gamma_{i_0} = (\gamma_d + \gamma_{i_0})\beta_{i_0} + \gamma_{i_0}^2y - \gamma_1\gamma_dy.$$

Hence

$$\frac{\gamma_{i_0}\beta_d}{\gamma_d\beta_1} + 1 = \frac{(\gamma_d + \gamma_{i_0})\beta_{i_0}}{\gamma_d\beta_1} + \frac{\gamma_{i_0}^2y}{\gamma_d\beta_1} - \frac{\gamma_1y}{\beta_1}.$$

We have  $\beta_1 = \gamma_{i_0}y\Psi_1$ . Therefore we have

$$\frac{|\gamma_{i_0}|^2y}{|\gamma_d\beta_1|} = \frac{|\gamma_{i_0}|}{|\gamma_d|} |\Psi_1|^{-1} \leq \lambda_0^2 \left| \frac{v_{i_0}}{v_d} \right|^a |\Psi_1|^{-1}$$

and

$$\frac{|\gamma_1|y}{|\beta_1|} = \frac{|\gamma_1|}{|\gamma_{i_0}|} |\Psi_1|^{-1} \leq \lambda_0^2 \left| \frac{v_1}{v_{i_0}} \right|^a |\Psi_1|^{-1}.$$

Finally, from

$$|\beta_{i_0}| \leq \frac{|m|}{y^{d-1}} |\gamma_1\Psi_{i_0}|$$

we deduce

$$\begin{aligned} \frac{(|\gamma_d| + |\gamma_{i_0}|)|\beta_{i_0}|}{|\gamma_d\beta_1|} &\leq (1 + \lambda_0^2) \left| \frac{\beta_{i_0}}{\beta_1} \right| \leq (1 + \lambda_0^2) \frac{|m|}{y^d} \frac{|\gamma_1\Psi_{i_0}|}{|\gamma_{i_0}\Psi_1|} \\ &\leq \lambda_0^2(1 + \lambda_0^2) \frac{|m|}{y^d} \left| \frac{v_1}{v_{i_0}} \right|^a \frac{|\Psi_{i_0}|}{|\Psi_1|}. \end{aligned}$$

Hence from the assumptions

$$|v_1| \leq \mu^{-1}|v_{i_0}| \quad \text{and} \quad |v_{i_0}| \leq \mu^{-1}|v_d|,$$

we deduce

$$\left| \frac{\gamma_{i_0}\beta_d}{\gamma_d\beta_1} + 1 \right| \leq 4|m|\lambda_0^4\mu^{-a} \frac{|\Psi_{i_0}|}{|\Psi_1|}.$$

The conclusion of (c) follows from (22).  $\square$

**Step 7.** (a) Assume  $|v_{i_0}| = |v_1|$ . Since  $v_{i_0} \in \mathbb{R}$ , we deduce from Lemma 11 that

$$|v_1| < |v_{d-1}|.$$

If  $|v_2| < |v_{d-1}|$ , then

$$\frac{|v_{d-1}|}{|v_{i_0}|} \geq \frac{|v_{d-1}|}{|v_2|} = \mu$$

and we are in the case (a) of the step 6.

If  $|v_2| = |v_{d-1}|$ , then  $i_0 = 1$ , we have

$$\frac{|v_{d-1}|}{|v_1|} \geq \mu$$

and again we are in the case (a) of the step 6.

(b) Assume  $|v_{i_0}| = |v_d|$ . Using Lemma 11, we deduce

$$|v_d| > |v_2|.$$

If  $|v_2| < |v_{d-1}|$ , then

$$\frac{|v_{i_0}|}{|v_1|} \geq \frac{|v_{d-1}|}{|v_2|} = \mu$$

and we are in the case (b) of the step 6.

If  $|v_2| = |v_{d-1}|$ , then  $i_0 = d$ , we have

$$\frac{|v_d|}{|v_2|} \geq \mu$$

and again we are in the case (b) of the step 6.

(c) Assume finally  $|v_1| < |v_{i_0}| < |v_d|$ . In particular we have  $2 \leq i_0 \leq d-1$ . Assume that we are neither in the case (a) nor in the case (b) of the step 6. From

$$\frac{|v_{d-1}|}{|v_{i_0}|} < \sqrt{\mu} \quad \text{and} \quad \frac{|v_{i_0}|}{|v_2|} < \sqrt{\mu}$$

we deduce

$$\frac{|v_{d-1}|}{|v_2|} < \mu.$$

Given the definition of  $\mu$ , it follows that we have  $|v_2| = |v_{d-1}|$ . Since  $v_{i_0}$  is real, Lemma 11 implies  $d = 3$  and therefore  $i_0 = 2$ ,  $|v_1| < |v_2| < |v_3|$  and

$$\mu = \min \left\{ \frac{|v_3|}{|v_2|}, \frac{|v_2|}{|v_1|} \right\}.$$

From

$$|\gamma_1| \leq \lambda_0 |v_1|^a \leq \lambda_0 \lambda^{-a/2} < 1, \quad |\beta_2| = |\beta_{i_0}| < 1$$

and

$$|\gamma_2 \beta_3| = |\gamma_2 \gamma_3 \Psi_3| y \geq \frac{y |\Psi_3|}{|\gamma_1|} \geq \lambda_0^{-1} \lambda^{a/2} |\Psi_3| > 1,$$

we deduce  $|\gamma_1 \beta_2| < 1 < |\gamma_2 \beta_3|$ , hence

$$\gamma_1 \beta_2 + \gamma_2 \beta_3 \neq 0.$$

There is an element in the Galois closure of the cubic field  $\mathbb{Q}(v)$  which maps  $v_1$  to  $v_2$ ,  $v_2$  to  $v_3$ ,  $v_3$  to  $v_1$ . Therefore,

$$\gamma_2 \beta_3 + \gamma_3 \beta_1 \neq 0.$$

From the part (c) of the step 6 we deduce

$$0 < \left| \frac{\gamma_2 \beta_3}{\gamma_3 \beta_1} + 1 \right| \leq 4m \lambda_0^4 \mu^{-a/2}.$$

**Step 8.** Combining the steps 6 and 7 with the step 4 where we choose

$$\begin{cases} i = \ell = d-1, j = k = d, c = 1 & \text{in the case (a),} \\ i = k = i_0, j = 1, \ell = d, c = 1 & \text{in the case (b),} \\ i = i_0, j = k = d, \ell = 1, c = -1 & \text{in the case (c).} \end{cases}$$

we deduce

$$\begin{aligned} a \log \mu &\leq \kappa_{32} R (R + \log |m| + \log \lambda_0 + \log \lambda) (\log \lambda) \\ &\quad \times \log \left( \frac{Ra \log \lambda}{R + \log |m| + \log \lambda_0 + \log \lambda} \right). \end{aligned}$$

For

$$U = \frac{Ra \log \lambda}{R + \log |m| + \log \lambda_0 + \log \lambda} \quad \text{and} \quad V = \kappa_{33} \frac{R^2 (\log \lambda)^2}{\log \mu},$$

we have  $U \leq V \log^* U$ . Therefore we use Lemma 4 to obtain the conclusion of Theorem 2.

## 5 Proof of Theorem 1

Since  $d \geq 3$ , under the assumptions of Lemma 12 we have

$$\log \frac{|v''|}{|v'|} \geq \frac{\kappa_{14}(\log \lambda)^2}{\lambda^{d^2(d+2)/2}}.$$

From Lemma 11, we deduce that under the assumptions of Theorem 1 and with the notations of Theorem 2, we have

$$\log \mu \geq \frac{\kappa_{34}(\log \lambda)^2}{\lambda^{d^2(d+2)/2}}.$$

Hence Theorem 2 implies Theorem 1.

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